JOURNAL OF DIFFERENTIAL EQUATIONS 1, 293-305 (1965)

Asymptotic Behavior of Nonlinear Delay-Differential Equations*

RICHARD K. MILLER

Department of Mathematics, University of Wisconsin, Madison, Wisconsin

1. INTRODUCTION

In this paper we study the behavior as $t \to \infty$ of solutions of certain non-linear delay-differential equations. We extend the notion of an invariant set to almost periodic equations and consider the interaction of this notion of invariance with perturbation theory and with the theory of Liapunov functions (cf. [*3-6*, *8*, *10*] for related results).

Let r be a fixed positive number and let D be a fixed open domain in n-space R^n. For any subset A of D let $C_r(A)$ be the space of all continuous functions mapping the interval $[-r, 0]$ into the set A. For any $\varphi \in C_r(A)$ define the norm of φ by

$$\| \varphi \| = \sup \{ | \varphi(u) |; -r \leqslant u \leqslant 0 \}.$$

Given any continuous function $x(t)$ from an interval $[T - r, L)$ into A, define the path x_t from $[T, L)$ into the space $C_r(A)$ by the formula

$$x_t(u) = x(t + u) \qquad \text{for} \qquad -r \leqslant u \leqslant 0.$$

DEFINITION. *A delay-differential equation is a relation of the form*

$$x'(t) = f(t, x_t), \qquad \left(' = \frac{d}{dt}\right) \tag{1.1}$$

where f is a continuous function from $I \times C_r(D)$ into D and $I = \{t; 0 \leqslant t < \infty\}$.

A solution of (1.1) with initial values (T, φ) is a continuous function $x(t)$ defined on an interval $[T - r, L)$ such that $x_T = \varphi$, $x_t \in C_r(D)$ for $T \leqslant t < L$, and $x(t)$ satisfies (1.1) for $T \leqslant t < L$.

We remark that if $f(t, \varphi)$ is continuous in the pair (t, φ), then for each (T, φ) there is a solution of (1.1). If f is locally Lipschitzian in φ, then solutions of (1.1) are uniquely determined to the right by their initial values. If for

* This work was partially supported by National Science Foundation (G-24335).

each compact subset $D^* \subset D$, $|f(t, \varphi)|$ is bounded on $I \times C_r(D^*)$, then each solution $x(t)$ of (1.1) can be continued to the right for as long as $x(t)$ remains in a compact subset of D. See Driver ([2], pp. 403-408) for proofs of the above statements.

In this paper we consider systems of n equations of the form

$$x'(t) = P(t, x_t), \tag{1.2}$$

and perturbations of the form

$$x'(t) = P(t, x_t) + R(t, x_t) + G(t, x_t), \tag{1.3}$$

where the following assumptions are made:

(H1) R and G are continuous on $I \times C_r(D)$ with values in D.

(H2) P is continuous on $R^1 \times C_r(D)$ and P is uniformly almost periodic in t.

(H3) For each compact subset $D^* \subset D$, $|P(t, \varphi)|$ is bounded on $R^1 \times C_r(D^*)$.

(H4) For any continuous function $y(t)$ defined on I with values in a compact set $D^* \subset D$, for any sequence $t_m \to \infty$ as $m \to \infty$ and any $t \in R^1$,

$$\operatorname*{limit}_{m\to\infty} \int_{t_m}^{t_m+t} |G(s, y_s)|\, ds = 0.$$

(H5) Let Q be a fixed subset of $C_r(D)$. For each compact set $D^* \subset D$ and each $\epsilon > 0$ there are fixed numbers $\delta > 0$ and $S > 0$ (depending only on ϵ and D^*) such that whenever $t \geqslant S$, $\varphi \in C_r(D^*)$ and $d(\varphi, Q) < \delta$ one has $|R(t, \varphi)| < \epsilon$.

We shall use the following conventions: if $y \in R^n$, then $|y|$ is any vector norm. The symbols $d(x, A)$ denote the distance from the point x to the set A. If $x(t)$ is any function and A any subset of R^n, $y(t) \to A$ as $t \to \infty$ if for each $\epsilon > 0$ there is a $t(\epsilon) > 0$ such that $d(y(t), A) < \epsilon$ for all $t \geqslant t(\epsilon)$. The definition of $y_t \to U$ as $t \to \infty$ is similar.

If $y(t)$ is a continuous function on an interval $[T, \infty)$ into D, the *positive limit set of* $y(t)$, denoted by $\Gamma^+(y(t))$, consists of all points z for which there is a sequence $t_m \to \infty$ as $m \to \infty$ with $y(t_m) \to z$. Similarly the positive limit set $\Lambda^+(y_t)$ consists of all functions $\varphi \in C_r(D)$ such that there is a sequence $t_m \to \infty$ as $m \to \infty$ with $\| \varphi - y_{t_m} \| \to 0$.

Remarks on Hypothesis (H2)

DEFINITION. *If $P(t, \varphi)$ is continuous on $R^1 \times C_r(D)$ with values in D, then $P(t, \varphi)$ is called uniformly almost periodic in t if and only if for each $\epsilon > 0$*

and each compact subset $U \subset C_r(D)$ there is a number $L = L(\epsilon, U) > 0$ such that in any interval $a < t < a + L$ of length L one can find a number t_0 such that

$$| P(t + t_0, \varphi) - P(t, \varphi) | < \epsilon \tag{1.4}$$

uniformly for all $t \in R^1$ and all $\varphi \in U$.

Existing real variable proofs are easily modified to establish the following results.

LEMMA 1. *If $P(t, \varphi)$ is uniformly almost periodic in t and if U is a compact subset of $C_r(D)$, then $| P(t, \varphi) |$ is bounded on $R^1 \times U$.*

LEMMA 2. *If $P(t, \varphi)$ is uniformly almost periodic in t and if $\{h_m\}$ is any real sequence, then there is a subsequence $\{h_{mk}\}$ and a uniformly almost periodic function $P^*(t, \varphi)$ such that*

$$P(t + h_{mk}, \varphi) \to P^*(t, \varphi) \qquad \text{as} \qquad k \to \infty \tag{1.5}$$

uniformly for all $t \in R^1$ and φ on compact subsets of $C_r(D)$.

If $P(t, \varphi)$ is almost periodic, let $H(P)$ be the set of functions

$$H(P) = \{P(t + h, \varphi); -\infty < t < \infty\}.$$

Let $\bar{H}(P)$ be the uniform closure, in the sense of (1.5), of $H(P)$. We shall use the space $\bar{H}(P)$ in the sequel.

Remarks on Hypothesis (H4)

The author is indebted to C. C. Conley for the present form of (H4). We remark that if the function $| G(t, y_t) | \in L_p(0, \infty)$ for some p in the interval $1 \leqslant p < \infty$, then (H4) is satisfied. For $p = 1$ this proof is trivial. If $p > 1$, then from the Hölder inequality we see that

$$\left| \int_{t_m}^{t_m+t} 1 \cdot | G(s, y_s) | \, ds \right| \leqslant | t |^{1-(1/p)} \cdot \left| \int_{t_m}^{t_m+t} | G(s, y_s) |^p \, ds \right|^{1/p}.$$

2. PERTURBED ALMOST PERIODIC SYSTEMS

A. *The Main Result*

THEOREM 1. *Let hypotheses* (H1)-(H5) *hold for some fixed set $Q \subset C_r(D)$. Let $x(t)$ be a solution of (1.3) on $T \leqslant t < \infty$ with the range of $x(t)$ in a compact set $D^* \subset D$. If $x_t \to Q$ as $t \to \infty$, then for each point $z \in \Gamma^+(x(t))$ there corres-*

ponds a sequence $t_m \to \infty$ *as* $m \to \infty$, *a function* $y(t)$ *and a function* $P^*(t, \varphi)$ *such that*

(a) $y(0) = z$ *and* $y'(t) = P^*(t, y_t)$ *for* $-\infty < t < \infty$,

(b) $x(t + t_m) \to y(t)$ *as* $m \to \infty$ *uniformly on compact subsets of* $t \in R^1$, *and*

(c) $P(t + t_m, \varphi) \to P^*(t, \varphi)$ *as* $m \to \infty$ *uniformly for all* t *and for* φ *on compact subsets of* $C_r(D)$.

Proof. Fix any point $z \in \Gamma^+(x(t))$. Pick any $L > 0$ and any decreasing' positive, null sequence $\{\epsilon_m\}$. From the definition of $\Gamma^+(x(t))$ there exists a sequence $t_m \to \infty$ as $m \to \infty$ with $x(t_m) \to z$. We may assume that

$$T + 2r + L < t_1 \qquad \text{and} \qquad t_m < t_{m+1} \qquad \text{for} \qquad m = 1, 2, 3, \cdots. \tag{2.1}$$

From (H4) it follows that we may assume

$$\int_{t_m-L-r}^{t_m+L} | G(s, x_s) | \, ds < \epsilon_m \qquad (m = 1, 2, 3, \cdots). \tag{2.2}$$

From (H5) it follows that there are numbers $S_m > 0$ and $\delta_m > 0$ such that whenever $t \geqslant S_m$, $\varphi \in C_r(D^*)$ and $d(\varphi, Q) < \delta_m$ one has $| R(t, \varphi) | < \epsilon_m$. Since $x_t \to Q$ as $t \to \infty$, the numbers S_m may be taken so large that $d(x_t, Q) < \delta_m$ for all $t \geqslant S_m$. We assume that $t_m \geqslant S_m + r + L$ and accordingly that

$$| R(t, x_t) | < \epsilon_m \qquad \text{for all} \qquad t \geqslant t_m - r - L. \tag{2.3}$$

Let M be the number defined by

$$M = \sup \{ | P(t, \varphi) |; -\infty < t < \infty, \varphi \in C_r(D^*) \}. \tag{2.4}$$

For any $t > T$ and $t_0 > T$ note that

$$x(t) = x(t_0) + \int_{t_0}^{t} \{ P(s, x_s) + R(s, x_s) + G(s, x_s) \} \, ds. \tag{2.5}$$

Define $x_m(t) = x(t + t_m)$ for $-r - L \leqslant t \leqslant L$ and $m = 1, 2, 3, \cdots$. Using (2.1) and (2.5) we see that

$$x_m(t) = x(t_m) + \int_0^t P(s + t_m, (x_m)_s) \, ds + \int_{t_m}^{t+t_m} (R(s, x_s) + G(s, x_s)) \, ds. \tag{2.6}$$

The sequence $\{x_m(t)\}$ is uniformly bounded by the compact set D^*. This sequence is also equicontinuous since for any m and for

$$-r - L \leqslant s_1 < s_2 \leqslant L,$$

it follows from (2.2)-(2.4) and (2.6) that

$$| x(s_1) - x(s_2) | \leqslant \int_{s_1}^{s_2} M ds + \int_{s_1}^{s_2} \epsilon_m ds + \int_{t_m-L-r}^{t_m+L} | G(s, x_s) | ds$$

$$\leqslant (M + \epsilon_m)(s_2 - s_1) + \epsilon_m .$$

This argument also shows that the set

$$U_0 = \{(x_m)_t; -L \leqslant t \leqslant L, m = 1, 2, 3, \cdots\}$$

has compact closure in $C_r(D)$.

There is a subsequence of $\{x_m(t)\}$ which we again by m and there is a function $y(t)$ such that

$$| x_m(t) - y(t) | = | x(t + t_m) - y(t) | \to 0 \tag{2.7}$$

as $m \to \infty$ uniformly for $-r - L \leqslant t \leqslant L$. Using Lemma 2 above we may assume for the same subsequence and for some $P^*(t, \varphi) \in \bar{H}(P)$ that

$$| P(t + t_m , \varphi) - P^*(t, \varphi) | \to 0 \qquad \text{as} \qquad m \to \infty. \tag{2.8}$$

Note that for any t in the interval $-L \leqslant t \leqslant L$ one has

$$\left| \int_0^t \{P(s + t_m , (x_m)_s) - P^*(s, y_s)\} ds \right|$$

$$\leqslant \int_0^t | P(s + t_m , (x_m)_s) - P^*(s, (x_m)_s) | ds + \int_0^t | P^*(s, (x_m)_s) - P^*(s, y_s) | ds. \tag{2.9}$$

Since

$$U_0 = \{(x_m)_t; -L \leqslant t \leqslant L, m = 1, 2, 3, \cdots\}$$

has compact closure, it follows from (2.8) that the first term on the right side of (2.9), tends to zero as $m \to \infty$. Since $P^*(t, \varphi)$ is uniformly continuous on compact sets of (t, φ), the second expression also tends to zero. Noting (2.2), (2.3), and (2.6)-(2.9) one sees that

$$y(t) = z + \int_0^t P^*(s, y_s) ds, \tag{2.10}$$

for $-L \leqslant t \leqslant L$.

For $L = 1$ pick a sequence $\{t_{1m}\}$ such that (2.10) is satisfied on $-1 \leqslant t \leqslant 1$. Pick a subsequence $\{t_{2m}\}$ of $\{t_{1m}\}$ such that (2.10) is satisfied on $-2 \leqslant t \leqslant 2$. The same function P^* will be obtained in both cases and the two functions $y(t)$ will coincide where both are defined. Continue for

$L = 3, 4, 5, \cdots$. The sequence $\{t_m\}$ which satisfies (a), (b), and (c) may be chosen as $t_m = t_{mm}$. This completes the proof of Theorem 1.

COROLLARY 1. *Let the hypotheses of Theorem 1 hold. Then the positive limit set $\Lambda^+(x_t)$ of the path x_t is not empty and for each element $\psi \in \Lambda^+(x_t)$ there is a sequence $t_m \to \infty$ as $m \to \infty$ and functions $y(t)$ and $P^*(t, \varphi)$ such that*

(d) *$y'(t) = P^*(t, y_t)$ on $-\infty < t < \infty$ with $y_0 = \psi$, and*

(e) *$\| x_{t+t_m} - y_t \| \to 0$ as $m \to \infty$ uniformly on compact subsets of $-\infty < t < \infty$.*

Proof. We know that $\Gamma^+(x(t))$ is not empty. If z is any point of $\Gamma^+(x(t))$, then from Theorem 1 there is corresponding function $y(t)$. It is easy to see that for each $t \in R^1$ the element $y_t \in \Lambda^+(x_t)$, that is, $\Lambda^+(x_t)$ is not empty.

Given any element $\psi \in \Lambda^+(x_t)$ there is a sequence $t_m \to \infty$ as $m \to \infty$ with $\| x_{t_m} - \psi \| \to 0$. Thus $x(t_m) \to \psi(0)$ as $m \to \infty$. Letting $z = \psi(0)$ it is clear that (d) and (e) follow from parts (a) and (b) of Theorem 1. This completes the proof of Corollary 1.

B. *Invariant Sets*

The conclusion of Corollary 1 suggests an invariance property of the set $\Lambda^+(x_t)$ which we define as follows:

DEFINITION. *If U is any subset of $C_r(D)$, then U is called quasi-invariant with respect to the almost periodic system $x'(t) = P(t, x_t)$ if and only if for each element $\psi \in U$ there corresponds an almost periodic function $P^*(t, \varphi) \in \bar{H}(P)$ and a solution $y(t)$ of $y'(t) = P^*(t, y_t)$ such that $y_0 = \psi$ and y_t exists and remains in a compact subset of U for $-\infty < t < \infty$.*

The conclusion of Corollary 1 is that $\Lambda^+(x_t)$ is a nonempty, quasi-invariant set with respect to system (1.2).

COROLLARY 2. *Let* (H1)-(H5) *hold for a fixed set Q. Let Q_0 be the largest quasi-invariant subset of Q with respect to (1.2). If $x(t)$ is any solution of (1.3) with range in a compact subset D^* of D, then $x_t \to Q_0$ as $t \to \infty$.*

Proof. Using Theorem 1 it is easily proved that $x_t \to \Lambda^+(x_t)$ as $t \to \infty$. From Corollary 1 it follows that $\Lambda^+(x_t) \subset Q_0$. Therefore $x_t \to Q_0$ as $t \to \infty$. This proves Corollary 2.

C. *The Special Case $P(t, \varphi)$ Periodic in t*

If $P(t, \varphi)$ is periodic in t of period $\theta > 0$, then the space $\bar{H}(P)$ is just the set $\{P(t + h, \varphi); 0 \leqslant h < \theta\}$. If $t_m \to \infty$ as $m \to \infty$ is any sequence, then by possibly taking a subsequence we may assume that $t_m = h + K_m\theta + s_m$, where $0 \leqslant h \leqslant \theta$, $\{K_m\}$ is an increasing sequence of positive integers and $s_m \to 0$ as $m \to \infty$. Using these facts one can easily modify the proof of Theorem 1 to establish the following result.

THEOREM 2. *Let the hypothesis of Theorem 1 hold. Let $P(t, \varphi)$ be periodic in t of period θ. Then for each $z \in \Gamma^+(x(t))$ there is an increasing sequence $\{K_m\}$ of positive integers, a number h with $0 \leqslant h < \theta$, and a function $y(t)$ such that*

(a$_1$) *$y(h) = z$ and $y'(t) = P(t, y_t)$ for $-\infty < t < \infty$, and*

(b$_1$) *$x(t + K_m\theta) \to y(t)$ as $m \to \infty$ uniformly on compact subsets $-\infty < t < \infty$.*

3. LIAPUNOV THEORY

In this section we apply the results of Section 2 to the system (1.2). The principle results of this section generalize certain results of Hale [3] and Krasovskiĭ [4]. If $W(t)$ is any function, then let

$$\dot{W}(t) = \limsup_{h \to 0^+} \frac{1}{h} (W(t + h) - W(t)).$$

For any number h with $-r \leqslant h \leqslant 0$ define the pseudo norm $\| \varphi \|_{[h,0]}$ on the space $C_r(D)$ by

$$\| \varphi \|_{[h,0]} = \sup \{| \varphi(u) |; h \leqslant u \leqslant 0\}.$$

The following result is easily proved by modifying some results of Hale [3].

LEMMA 3. *Let the function $f(t, \varphi)$ of the system (1.1) $x'(t) = f(t, x_t)$ satisfy* (H3) *and let $V(t, \varphi)$ be a continuous function on $I \times C_r(D)$. Define*

$$U(L) = \{(t, \varphi); V(t, \varphi) < L\}.$$

Suppose there are constants h and K such that $-r \leqslant h \leqslant 0$,

$$\bar{N} = \{y; | y | \leqslant K\} \subset D,$$

and for all $(t, \varphi) \subset U(L)$ one has $\| \varphi \|_{[h,0]} \leqslant K$ and $V(t, \varphi) \geqslant 0$. If $\dot{V}(t, x_t) \leqslant 0$ for all solutions x_t of (1.1), then all solutions of (1.1) which start in the set $U(L)$ exist and are bounded for all future time.

Proof. Let $x(t)$ be a solution of (1.1) with initial values $(T, x_T) \in U(L)$. Since $\dot{V}(t, x_t) \leqslant 0$ we see that $V(t, x_t) < L$ and $x(t) \in \bar{N}$ for as long as $x(t)$

exists. From the compactness of $\bar{N}$ and (H3) it follows that $x(t)$ exists and remains in $\bar{N}$ for all $t \geqslant T$. This proves the lemma.

The main result of this section may be stated as follows:

THEOREM 3. *Suppose $V \in C(R^1 \times C_r(D))$ with $\dot{V}(t, x_t) \leqslant 0$ for each solution $x(t)$ of (1.2). Suppose $P(t, \varphi)$ and $V(t, \varphi)$ are uniformly almost periodic in t and $P(t, \varphi)$ satisfies* (H3). *Let U be the subset of $C_r(D)$ consisting of all trajectories y_t such that corresponding to y_t there exist $P^* \in \bar{H}(P)$ and $V^* \in \bar{H}(V)$ such that*

(i) $y'(t) = P^*(t, y_t)$ *for* $-\infty < t < \infty$.

(ii) $y(t)$ *has range in a compact set* $D_0 \subset D$, *and*

(iii) $\dot{V}^*(t, y_t) = 0$ *for* $-\infty < t < \infty$.

If $x(t)$ is a solution of (1.2) which exists for all large t and has range in a compact set $D^* \subset D$, then $x_t \to U$, as $t \to \infty$.

Proof. We shall show that the positive limit set $\Lambda^+(x_t)$ is contained in U. This will prove the theorem.

Since the range of $x(t)$ is in the compact set D^*, it follows from (H3) that the set

$$X = \{x_t;\, t \text{ sufficiently large}\}$$

is equicontinuous and uniformly bounded. Since X has compact closure in $C_r(D)$, it follows from Lemma 1 above that $|V(t, x_t)|$ is bounded. Together with $\dot{V}(t, x_t) \leqslant 0$ this implies that limit $V(t, x_t) = V_0$ as $t \to \infty$ exists.

Note that system (1.2) is a special case of (1.3) with $R(t, \varphi) \equiv G(t, \varphi) \equiv 0$ so that Theorem 1 applies to (1.2). For any fixed $\psi \in \Lambda^+(x_t)$ pick a sequence $t_m \to \infty$ as $m \to \infty$ such that $x_{t_m} \to \psi$, $V(t + t_m, \varphi) \to V^*(t, \varphi)$ and $P(t + t_m, \varphi) \to P^*(t, \varphi)$. The corresponding path y_t given by Corollary 1 above clearly satisfies (i) and (ii) of this theorem. Since $V^*(t, y_t) \equiv V_0$ constant, (iii) is also satisfied. Therefore $\Lambda^+(x_t) \subset U$ and Theorem 3 is proved.

In case P and V are periodic in t Theorem 3 takes the following form.

COROLLARY 3. *Let $P(t, \varphi)$ and $V(t, \varphi)$ be periodic in t of the same period θ. Suppose $P(t, \varphi)$ satisfies* (H3) *and $\dot{V}(t, y_t) \leqslant 0$ for each solution $y(t)$ of (1.2). Let U be the set of solutions y_t such that $y(t)$ has compact range and $\dot{V}(t, y_t) = 0$ for $-\infty < t < \infty$. If $x(t)$ is a solution of (1.2) with compact range, then $x_t \to U$ as $t \to \infty$.*

By combining Corollary 3 and Lemma 3 we can generalize certain results of Hale ([*3*], Theorem 1) and Krasovskiĭ ([*4*], p. 153) with the following result.

COROLLARY 4. *Let $P(t, \varphi)$ be periodic in t of period θ and let P satisfy* (H3). *Suppose $V(t, \varphi)$ is continuous in (t, φ) and periodic in t of period θ. Let there exist numbers L, h, and K such that*

(i) $-r \leqslant h \leqslant 0$,

(ii) $\bar{N} = \{y \in R^n; |y| \leqslant K\} \subset D$,

(iii) $V(t, \varphi) \geqslant 0$ *and* $\|\varphi\|_{[h,0]} \leqslant K$ *for each* (t, φ) *in* $U(L)$, *and*

(iv) $\dot{V}(t, y_t) \leqslant 0$ *for all solutions* y_t *of* (1.2).

If U_0 is the set of solutions y_t such that $|y(t)| \leqslant K$ and $\dot{V}(t, y_t) = 0$ on $-\infty < t < \infty$, then all solutions $x(t)$ of (1.2) *with initial values in $U(L)$ remain in $\bar{N}$ and $x_t \to U_0$ as $t \to \infty$.*

An instability result of Krasovskiĭ ([4], p. 69) for ordinary differential equations admits the following extension:

COROLLARY 5. *Let $P(t, \varphi)$ and $V(t, \varphi)$ be periodic in t of period θ. Let P satisfy* (H3). *Suppose*

(i) *$\dot{V}(t, x_t) \geqslant 0$ for each solution x_t of (1.2),*

(ii) *for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $\|\varphi\| < \delta$, then $|V(t, \varphi)| < \epsilon$ uniformly for $t \in R^1$, and*

(iii) *there is no solution $y(t)$ of (1.2) such that $y(t) \not\equiv 0$, $y(t)$ has compact range and $\dot{V}(t, y_t) = 0$ on $-\infty < t < \infty$. If $x(t)$ is a solution of (1.2) with $V(T, x_T) > 0$ for some $T > 0$, then for each compact subset $D^* \subset D$ there is a number $t_1 = t_1(D^*, x(t)) \geqslant T$ such that $x(t_1) \notin D^*$.*

Proof. Suppose there is a compact set D^* such that $x(t) \in D^*$ for all $t \geqslant T$. From Theorem 3 and (iii) above we see that zero must be a solution of (1.2) and $x_t \to 0$ as $t \to \infty$. From $\dot{V}(t, x_t) \geqslant 0$ it follows that $V(t, x_t) \geqslant V(T, x_T)$ for all $t \geqslant T$. From (ii) there is a $\delta > 0$ such that if $\|x_t\| < \delta$, then $|V(t, x_t)| < V(T, x_T)$. Thus $x_t \to 0$ and $\|x_t\| \geqslant \delta$, a contradiction which proves Corollary 5.

Corollary 5 is not quite an instability result in its present form. If it is also assumed that $x(t) \equiv 0$ is a solution of (1.2) and that for each positive integer m there exists $t_m \geqslant m$ and φ_m with $\|\varphi_m\| \leqslant 1/m$ and $V(t_m, \varphi_m) > 0$, then it is easy to see that $x(t) \equiv 0$ is not stable.

4. EXAMPLES

As an application of the results of the previous sections we consider the behavior as $t \to \infty$ of solutions of the equation

$$x'(t) = -\frac{1}{r}\int_{-r}^{0} (r + u)\, g(x(t + u))\, du, \tag{4.1}$$

where $r > 0$ and $g(x) \in C(-\infty, \infty)$. If $x(t)$ is a solution of (4.1), then with $y(t) = x'(t)$ it is easy to see that $(x(t), y(t))$ is a solution of

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -g(x(t)) + \frac{1}{r}\int_{-r}^{0} g(x(t+u))\,du. \end{aligned} \tag{4.2}$$

On the other hand, if $(x(t), y(t))$ is a solution of (4.2), then $x(t)$ is a solution of

$$\begin{aligned} x'(t) &= c - \frac{1}{r}\int_{-r}^{0} (r+u)\,g(x(t+u))\,du, \\ &= -\frac{1}{r}\int_{-r}^{0} (r+u)\left(g(x(t+u)) - \frac{2c}{r}\right)du, \end{aligned} \tag{4.3}$$

where

$$c = y(0) + \frac{1}{r}\int_{-r}^{0} (r+u)\,g(x(u))\,du.$$

Our analysis of (4.1) and (4.2) will depend heavily on the work of Levin and Nohel [7]. We introduce the Liapunov function

$$V(x_t) = G(x(t)) + \frac{1}{2r}\int_{-r}^{0}\left[\int_{\tau}^{0} g(x(t+u))\,du\right]^2 d\tau$$

where x_t is a solution of (4.1) and

$$G(x) = \int_0^x g(u)\,du.$$

A calculation involving integration by parts yields

$$\dot V(x_t) = -\frac{1}{2r}\left[\int_{-r}^{0} g(x(t+u))\,du\right]^2.$$

Lemma 4. *For each bounded solution $x(t)$ of (4.1), one has*

$$\operatorname*{limit}_{t\to\infty} \int_{-r}^{0} g(x(t+u))\,du = 0.$$

Proof. Since

$$\dot V(x_t) = -\frac{1}{2r}\left(\int_{-r}^{0} g(x(t+u))\,du\right)^2,$$

Lemma 4 follows by an application of Theorem 3.

Levin and Nohel [7] have proved the following result in the special case where $g(x)$ is locally Lipschitzian.

THEOREM 4. *Let the hypothesis of Lemma 4 hold. Suppose* $xg(x) > 0$ *for* $x \neq 0$ *and suppose* $G(x) \to \infty$ *as* $|x| \to \infty$. *Let* $\varphi \in C_r(R^1)$ *be arbitrarily given. Then the following statements are true:*

(i) *There is a solution* $x(t)$ *of (4.1) which is continuous on* $-r \leqslant t < \infty$ *with* $x(t) = \varphi(t)$ *on* $-r \leqslant t \leqslant 0$.

(ii) $|x^{(j)}(t)|$ *is bounded on* $0 \leqslant t < \infty$ *for* $j = 0, 1, 2$.

(iii) *The positive limit set* $\Gamma^+(x(t), x'(t))$ *is a single closed orbit of the system*

$$u'(t) = v(t), \qquad v'(t) = -g(u(t)). \tag{4.4}$$

(iv) *If* $\Gamma^+(x(t), x'(t)) \neq (0, 0)$ *and if* p *is the least common period of the solutions of (4.4) which generate* $\Gamma^+(x(t), x'(t))$, *then* $r = mp$ *for some integer* $m \geqslant 1$.

Proof. Statements (i) and (ii) follow exactly as in [7]. They may also be proved from Lemma 3 above. With $G(t, \varphi) \equiv 0$,

$$P(\varphi, \psi) = \begin{pmatrix} \psi(0) \\ -g(\varphi(0)) \end{pmatrix}, \qquad R(\varphi, \psi) = \begin{pmatrix} 0 \\ \dfrac{1}{r} \displaystyle\int_{-r}^{0} g(\varphi(u))\, du \end{pmatrix},$$

and $Q = \{\varphi;\, R_2(\varphi, \psi) = 0\}$ we apply Theorem 2 to system (4.2). Thus if $(a, b) \in \Gamma^+(x(t), x'(t))$ and if $(u(t, t_0, a, b), v(t, t_0, a, b))$ is the solution of (4.4) which passes through the point (a, b) at $t = t_0$, then there are sequences $t_m \to \infty$ as $T_m \to \infty$ as $m \to \infty$ such that

$$\sup_{-T_m \leqslant t \leqslant T_m} |x(t + t_m) - u(t, 0, a, b)| \to 0$$

and

$$\sup_{-T_m \leqslant t \leqslant T_m} |x'(t + t_m) - v(t, 0, a, b)| \to 0 \qquad \text{as} \qquad m \to \infty.$$

Also,

$$\int_{-r}^{0} g(u(t + s, 0, a, b))\, ds = 0 \qquad \text{for} \qquad -\infty < t < \infty.$$

The proof may now be completed in exactly the same manner as the proof given by Levin and Nohel ([7], pp. 40-43).

COROLLARY 6. *Let the hypotheses of Theorem 4 hold. Let* $g(x)$ *be strictly increasing with* $|g(x)| \to \infty$ *as* $|x| \to \infty$. *If* $\varphi, \psi \in C_r(R^1)$ *are any initial values, then the following results are true:*

(i) *There is a solution* $(x(t), y(t))$ *of (4.2) defined, continuous and bounded on* $-r \leqslant t < \infty$ *with* $x_0 = \varphi$ *and* $y_0 = \psi$.

(ii) *There exist numbers* x^* *and* y^* *such that the set* $\Gamma^+(x(t) - x^*, y(t))$ *is a single closed orbit of*

$$u'(t) = v(t), \qquad v'(t) = -g(u + x^*) + y^*. \tag{4.5}$$

(iii) *If* $\Gamma^+(x(t) - x^*, y(t)) \neq (0, 0)$ *and if* p *is the least common period of the periodic solutions which generate* $\Gamma^+(x(t) - x^*, y(t))$, *then* $r = mp$ *for some integer* $m \geqslant 1$.

Proof. System (4.2) with initial values (φ, ψ) is equivalent to system (4.3) with

$$c = \psi(0) + \frac{1}{r}\int_{-r}^{0} (r + u)\, g(\varphi(u))\, du.$$

Let $y^* = 2c/r$ and x^* be the unique point where $g(x^*) = y^*$. Let

$$w(t) = x(t) - x^* \qquad \text{and} \qquad f(w) = g(w + x^*) - y^*.$$

Then (4.3) is transformed to the equation

$$w'(t) = \frac{1}{r}\int_{-r} (r + u) f(w(t + u))\, du, \tag{4.6}$$

where $f(0) = 0$, $wf(w) > 0$ if $w \neq 0$ and

$$\int^{w} f(u)\, du \to \infty \qquad \text{as} \qquad w \to \infty.$$

If we apply Theorem 4 to (4.6) and then use the transformation $x(t) = w(t) + x^*$ we see that Corollary 6 is established.

We remark that the hypotheses of Theorem 4 are not generally sufficient to establish even the boundedness of solutions of (4.2). For example $g(x) = \exp(x) - 1$ and φ and ψ are picked so that

$$\psi(0) + \frac{1}{r}\int_{-r}^{0} (r + u)\, g(\varphi(u))\, du < -\frac{r}{2},$$

then a consideration of (4.3) and Lemma 4 shows that the solution of (4.2) with initial values (φ, ψ) is not bounded as t increases to some finite or infinite limit value.

Acknowledgments

The author wishes to thank Professor J. A. Nohel and C. C. Conley for their many helpful suggestions.

References

1. Besicovitch, A. S. "Almost Periodic Functions." Dover, New York, 1954.
2. Driver, R. D. Existence and stability of solutions of a delay-differential system. *Arch. Rat. Mech. Anal.* **19** (1962), 401-426.
3. Hale, J. K. A stability theorem for functional-differential equationI. *Proc. Nat. Acad. Sci.* **50** (1963), 942-946.
4. Karsovskiĭ, N. N. "Stability of Motion" (English translation by J. B. Brenner). Stanford Univ. Press, Stanford, Calif., 1963.
5. LaSalle, J. P. Asymptotic stability criterion. *Proc. Symp. Appl. Math.* **13** (1962), 299-307.
6. Levin, J. J. On the global asymptotic behavior of nonlinear systems of differential equations. *Arch. Rat. Mech. Anal.* **6** (1960), 65-74.
7. Levin, J. J., and Nohel, J. A. On a nonlinear delay equation. *J. Math. Anal. Appl.* **8** (1964), 31-44.
8. Miller, R. K. Asymptotic behavior of solutions of nonlinear differential equations. *Trans. Am. Math. Soc.*, in press.
9. Nohel, J. A. Problems in qualitative behavior of solutions of nonlinear Volterra equations. "Nonlinear Integral Equations." Univ. of Wisconsin Press, Madison, Wisconsin, 1964.
10. Yoshizawa, T. Asymptotic behavior of solutions of a system of differential equations. *Contrib. Diff. Eqs.* **1** (1963), 371-387.